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PARTIAL ATTRACTION, SEMI-STABLE LAWS, AND ITERATED LOGARITHM RESULTS: A SURVEY

GOOTY DIVANJI [1] | KABELO MOSEKI [2]

[1] FLAT NO: 202, SANKALP TEMPLE TREES APARTMENT, RMP LAYOUT, 2ND PHASE, VIJAYANAGAR 4TH STAGE, MYSORE - 560032, INDIA.

[2] DEPARTMENT OF STATISTICS, UNIVERSITY OF BOTSWANA, PRIVATE BAG UB 00704, GABORONE – BOTSWANA.

ABSTRACT:

The work surveys stable, semi-stable, and infinitely divisible laws, emphasizing domains of attraction and partial attraction. It highlights key results on limit theorems, especially laws of the iterated logarithm under power normalisation. Contributions include extensions to stable and semi-stable domains, subsequences, delayed sums, random sums, and related stochastic processes.

KEYWORDS:

STABLE, SEMI-STABLE, INFINITELY DIVISIBLE LAWS AND DOMAIN OF PARTIAL ATTRACTION.

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INTRODUCTION:

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables, all having the same distribution function F, and defined on the same probability space $(\Omega, \Im, P)$. Let $S_n = \sum_{k=1}^{n} X_k$, for $n \geq 1$.

This sequence $\{S_n\}$ is called the **partial sum sequence**.

In probability theory, there are three major limit theorems:

I. The Law of Large Numbers

II. The Central Limit Theorem

III. The Law of the Iterated Logarithm

These theorems are based on the concept of weak convergence or are derived from the central limit theorem. Before we discuss them briefly, let's review some important terms that will be used.

SEQUENCE OF INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES

We say that $\{X_n, n \geq 1\}$ is a sequence of **independent random variables** if any group of these variables, no matter how small or large, is all independent of each other. Let **X** and **Y** be two random variables with distribution functions **F** and **G**, respectively. We say that **X** and **Y** are **identically distributed** if they follow the same distribution; that is, **F(x) = G(x)** for all real values of x.

A sequence $\{X_n, n \geq 1\}$ is called **independent and identically distributed (i.i.d.)** if:

The variables are independent of each other, and each random variable X_n has the same

distribution as the first one, meaning $\mathbf{P(X_n \leq x) = F(x)}$ for all $n \geq 2$.

MODES OF CONVERGENCE

There are different ways in which a sequence of random variables can approach a limit. These are called **modes of convergence**. We'll look at three types of convergence for random variables, and a kind for distributions.

1. CONVERGENCE IN PROBABILITY

A sequence of random variables $\{X_n\}$ is said to **converge in probability** to a random variable X if, for every small number $\varepsilon > 0$, $\mathbf{P(|X_n - X| > \varepsilon) \to 0 \text{ as } n \to \infty}$

This means that the probability that X_n is far from X becomes smaller and smaller as n increases. We write this as: **$X_n \to X$ in probability**, or symbolically: $X_n \xrightarrow{P} X$.

2. ALMOST SURE CONVERGENCE

A sequence $\{X_n\}$ **converges almost surely** (also called **with probability 1**) to a random variable X if: $P\left(\lim_{n \to \infty} X_n = X\right) = 1$. In simpler terms, this means that with probability 1, the values of X_n will eventually settle down and stay close to X forever. We write this as:

$X_n \to X$ almost surely, or symbolically: $X_n \xrightarrow{a.s} X$.

3. CONVERGENCE IN DISTRIBUTION

A sequence of random variables $\{X_n, n \geq 1\}$, with corresponding distribution functions $\{F_n\}$, is said to **converge in distribution** (or **weakly**) to a random variable **X** with distribution function **F** if: $\mathbf{F_n(x) \to F(x)}$ as $\mathbf{n \to \infty}$, at every point where **F** is continuous. This type of convergence is written as: $\mathbf{X_n \to X}$ **in distribution**, or

$X_n \xrightarrow{d} X$, $n \to \infty$.

I. LAWS OF LARGE NUMBERS

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with a common distribution function. **F**, defined on the same probability space $(\Omega, \Im, P)$.Let the partial sum sequence be:

$$S_n = \sum_{k=1}^{n} X_k \text{, for } n \geq 1.$$

We are interested in knowing **under what conditions** on $\{X_n\}$ there exists a constant random variable $\boldsymbol{\xi}$ (called **degenerate**, because it takes only one value), such that: $S_n / n \to \xi$, under any mode of convergence.

If this happens with convergence in probability, we say the **Weak Law of Large Numbers (WLLN)**hold.

If it happens under **almost sure convergence**, we say the **Strong Law of Large Numbers (SLLN)** hold.

Let's look at some examples:

(A) KHINTCHINE'S WEAK LAW OF LARGE NUMBERS (WLLN)

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. If the expected value $\mathbf{E(X_1) = \mu}$ (a finite mean exists), then: $\mathbf{S_n / n \to \mu}$ **in probability**, as $\mathbf{n \to \infty}$.This means that the sample average gets closer and closer to the expected value μ with high probability as the number of terms increases.

(B) KOLMOGOROV'S WEAK LAW OF LARGE NUMBERS (WLLN):

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common distribution function F. A necessary and sufficient condition for the Weak Law of Large Numbers to hold for $\frac{S_n}{n} - \xi_n \xrightarrow{P} 0$ is that $\lim_{t \to \infty} t\{1-F(t)+F(-t)\} = 0$.

If the condition is satisfied ξ_n may be taken as $\xi_n = \int_{-n}^{n} x\, dF(x)$.

Example 1: Consider a random variable X with the distribution function

$$F(x)=\begin{cases} 0, & \text{if } x<a_0 \\ 1-\dfrac{1}{x \log x}, & \text{if } \geq a_0 \end{cases}$$

, where a_0 is a solution of $a_0 \log a_0=1$.

This example does not satisfy Khintchine's WLLN but does satisfy Kolmogorov's WLLN.

Example 2: Consider the standard Cauchy distribution. Since E(X) does not exist, Khintchine's WLLN does not hold. Kolmogorov's WLLN also fails in this case.

(C) TCHEBYCHEV'S WEAK LAW OF LARGE NUMBERS:

Let $\{X_n, n \geq 1\}$ be a sequence of uncorrelated random variables such that $E(X_n) = \mu_n$, $Var(X_n) = \sigma_n^2$,and$Cov(X_i, X_j) = 0$ for $i \neq j$.

Define the sample mean and the average mean as: $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$,and $\overline{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i$.

Then, $\bar{X}_n$ converges in probability to $\overline{\mu}_n$, as $n \to \infty$,provided that: $\frac{\sum_{k=1}^{n} \sigma_k^2}{n^2} \to 0$ as $n \to \infty$.

II. CENTRAL LIMIT THEOREM

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and let $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, be the sequence of partial sums.

The Central Limit Problem determines under what conditions on $\{X_n, n \geq 1\}$ there exist sequences of real constants $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ (with $B_n \to \infty$ as $n \to \infty$) such that the sequence $(S_n - A_n) / B_n$ converges (weakly) to a non-degenerate random variable (particularly, a normal variable).

Some important versions of the Central Limit Theorem (CLT) are presented below.

(D) LINDEBERG–FELLER CENTRAL LIMIT THEOREM

Let $\{X_n, n \geq 1\}$ be a sequence of independent, identically distributed random variables with $Var(X_n) = \sigma_n^2 < \infty$, n = 1, 2, 3, ...; Let $S_n = \sum_{i=1}^{n} X_i$, $E(X_n) = \mu_n$, $m_n = E(S_n)$ and $s_n^2 = Var(S_n)$.Then the following two conditions:

(i) $(S_n - m_n)/s_n$ converges in distribution to a standard normal variable, and

(ii) $E(\exp\{it(S_n - m_n)/s_n\}) \to \exp\{-t^2/2\}$ for every $t \in \mathbb{R}$, holds if and only if, for every $\varepsilon > 0$, the condition

(iii) $\left(\frac{1}{s_n^2}\right)\sum_{i=1}^{n} E\left((X_i-\mu_i)^2 .1\{|X_i-\mu_i|>\in s_n\}\right) \to 0$

is satisfied.

The sufficiency part of condition (iii) is called the Lindeberg condition, and the necessity part of condition (iii) is called the Feller condition. Together, they form the Lindeberg–Feller Central Limit Theorem (CLT).

(E) CENTRAL LIMIT THEOREM DUE TO PAUL LÉVY

(F) CENTRAL LIMIT THEOREM DUE TO LYAPUNOV

III. LAWOFTHE ITERATED LOGARTM:

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random

variables with common distribution function F, defined on a common probability space

$(\Omega, \Im, P)$. Let $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. The sequence $\{S_n\}$ is called the partial sum sequence. Set $Z_n = \frac{S_n}{B_n} - A_n$, $n \geq 1$, n ≥1, where $\{A_n\}$ and $\{B_n\}$ are some norming constants with $B_n>0$.

When $B_n = n$ and $A_n = 0$, the laws of large of numbers tells that the sequence $\{Z_n,\ n \geq 1\}$ converges to a degenerate random variable, under any modes of convergence, discussed above. i.e., convergence in probability or almost sure convergence.

Secondly, central limit problems tell that under what conditions on $\{X_n,\ n \geq 1\}$, there exist $\{A_n,\ n \geq 1\}$ and $\{B_n,\ n \geq 1\}$ be sequences of real constants ($B_n \to \infty$, as $n \to \infty$), such that a sequence $\{Z_n\}$ converges (weakly) to a non-degenerate random variable (particularly normal variable).

Therefore, it is natural to ask what happens in between the laws of large numbers and the central limit problem, when the limit does not exist, i.e., whether non-trivial limit behaviour is obtainable or not? Hence, the study relates to these types of issues, we call it as "Law of Iterated Logarithm". i.e., the studies of the behaviour of the upper sums limit and the lower sums limit is called "Law of Iterated Logarithm" (LIL). Hence, one wants to study whether Limit infimum or Limit suprimum exists for $Z_n = \frac{S_n}{B_n} - A_n$ or not?

In 1924, A. YA.Khinchine is the first person to study these types of problems. In fact, he developed this in number theory. Later, A. N. Kolmogorov [1929] developed the theory for bounded random variables with finite variance.

(G) KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM

Let$\{X_n, n \geq 1\}$ be a sequence of independent random variables with

$E(X_n) = 0$ and $E\left(X_n^2\right) = \sigma_n^2$, for all $n \geq 1$

$S_n = \sum_{k=1}^{n} X_k$ and $B_n^2 = \sum_{k=1}^{n} \sigma_k^2$ for $n \geq 1$.

Let $\{k_n\}$) be a sequence of positive constants such that $k_n \to 0$ as $n \to \infty$.If the following conditions hold:

1. $\frac{X_n}{B_n} \to 0$ as $n \to \infty$,
2. $\sum_{n=1}^{\infty} P\left(|X_n| > k_n B_n\right) < \infty$ a.s., then

$$\lim_{n \to \infty} \text{Sup} \frac{S_n}{\sqrt{2\, B_n^2\ \text{loglog} B_n^2}} = 1 \text{ a.s.}$$

Proof:

Set $Y_n = X_n 1_{\{|X_n| k_n B_n\}}$. Then it suffices to show that for every $\in > 0$,

$P\left(\left|\frac{S_n - E(Y_n)}{\sqrt{2B_n^2\ \text{loglog}\ B_n^2}}\right| > (1+\in)\ \text{i.o}\right) = 0$, and using the Borel–Cantelli lemma, one can prove this; details are omitted. If we replace every (X_n) by $(-X_n)$, we obtain the corresponding lower bound.

In the case where the (X_n) are unbounded but are independent and identically distributed random variables, Hartman and Wintner (1941) established that the existence of the second moment is indeed sufficient for this LIL for the partial sums. Their result is known as the Hartman–Wintner type LIL, or the classical law of the iterated logarithm.

(H) HARTMAN AND WINTNER LAW OF THE ITERATED LOGARITHM

Let $\{X_n\}$ be a sequence of independent and identically distributed random variables

with $E(X_1^2) < \infty$ and $EX_1 = 0$. Then $\lim_{n \to \infty} \text{Sup} \frac{S_n}{\sqrt{2n\ \text{loglog}\ n}} = 1$ a.s. The proof follows by using a truncation method together with the Borel–Cantelli lemma.

When $E\left(|X_1|^{2+\in}\right) < \infty$ Allan Gut (1986) established the classical LIL for geometrically fast increasing subsequences of the partial sums. In fact, he proved that

$$\lim_{k \to \infty} \text{Sup} \frac{S_{n_k}}{\sqrt{2B_{n_k}\ \text{loglog}\ B_{n_k}}} = \sqrt{1+\in^*} \text{ a.s.,}$$

where $\in^*$ is a constant determined by the growth rate of the subsequence $\{n_k\}$. Torrang (1987) extended the result to random subsequences.

Observe that when $\frac{n_{k+1}}{n_k} \to \infty$, then ($e^* = 0$), and we recover the classical result:

$$\lim_{k\to\infty} \text{Sup} \frac{S_{n_k}}{\sqrt{2B_{n_k} \text{ loglog } B_{n_k}}} = 1 \text{ a.s.}$$

That is, for such cases, the norming sequence $\sqrt{2B_{n_k}^2 \text{loglog} B_{n_k}^2}$ is not precise enough to give

almost-sure bounds for $\left(S_{n_k}\right)$.

In general, whenever $\frac{n_{k+1}}{n_k} \to \infty$ Rainer Schwabe and Allan Gut (1996) have shown that the classical normalizing sequence must be replaced by an adjusted one:

$$\sqrt{2B_{n_k}^2 \left(\text{loglog } B_{n_k}^2 + \text{loglog}\left(\frac{n_{k+1}}{n_k}\right)\right)}.$$

Note that sequences of the form ($n_k = [c^k]$, ($c > 1$), fall within the class of geometrically increasing subsequences.

For a comprehensive treatment of the LIL literature, see Bingham (1986).

WHEN THE SECOND MOMENT IS INFINITE

To understand the LIL in the case where the second moment is infinite, we first need to discuss certain types of distributions that possess infinite second moments.

INFINITELY DIVISIBLE DISTRIBUTIONS

A distribution function F is called *infinitely divisible* (i.d.) if, for every positive integer n, there exists a distribution function F_n such that $F = F_n^{*n}$, that is, F is the n-fold convolution of F_n.

Equivalently, a characteristic function ϕ is said to be infinitely divisible if, for every integer n, there exists a characteristic function ϕ_n such that $\phi(t) = \left(\phi_n(t)\right)^n$.

EXAMPLES

1. **NORMAL DISTRIBUTION:**

   Let ϕ be the characteristic function of a normal random variable with mean μ and variance σ^2. Then $\phi(t) = \exp\left\{i\mu t - \frac{1}{2}t^2\sigma^2\right\}$ which is clearly infinitely divisible.

2. **POISSON DISTRIBUTION:**

   For a Poisson distribution with parameter λ, $\phi(t) = \exp\left\{\lambda(e^{it}-1)\right\}$.

   and hence ϕ (t) is infinitely divisible.

3. **GAMMA DISTRIBUTION:**

   For a Gamma distribution with shape α and rate β, $\phi(t) = \left(1 - it/\beta\right)^{-\alpha}$. which is also infinitely divisible.

REMARK:

The class of all infinitely divisible distributions coincides with the class of limit distributions (in the sense of weak convergence) of sums of independent random variables. Moreover, the weak limit of a sequence of infinitely divisible distributions, if it exists, is itself infinitely divisible.

DISTRIBUTIONS OF CLASS L

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables, and let $\{b_n\}$ be a sequence of positive real numbers such that the following **uniform asymptotic negligibility** (U.A.N.) condition holds: $\underset{1\leq k\leq n}{\text{Max}}\, P\left(\left|X_k\right| > \varepsilon b_n\right) \to 0$, for every $\varepsilon > 0$.

Write $X_{n,k} = \frac{X_k}{b_n}$, $1 \leq k \leq n$. Then the triangular array $\{X_{n,k}\}$ satisfies the U.A.N. condition.

Set $S_n = \sum_{k=1}^{n} X_{n,k}$, $n \geq 1$. Let **Class L** be the set of all distributions that arise as weak limits of the distributions of the sums $a_n + b_n S_n$, where a_n and $b_n > 0$ are suitably chosen constants.

It follows that **Class L** forms a subclass of the infinitely divisible distributions.

A distribution function F with characteristic function ϕ belongs to **Class L, if and only if**, for every $0 < c < 1$, there exists a characteristic function ϕ, $\phi(t) = \phi(ct)\phi_c(t)$, $t \in \mathbb{R}$.

REMARK:

Degenerate distributions, the normal distribution, the Cauchy distribution, and the two-parameter Laplace distribution all belong to **Class L.**

STABLE DISTRIBUTIONS

A distribution function F is said to be stable if and only if, for every $b_1 > 0$, $b_2 > 0$, and real numbers a_1 and a_2, there existb > 0and a real number a such that

$F(b_1 x + a_1) * F(b_2 x + a_2) = F(bx + a)$,

or the equivalent defining relation your source implies.

The characteristic function ϕ (t) of a stable distribution has the following representation:

$$\phi(t) = \exp\left\{iat - c|t|^\alpha \left(1\text{-}i\beta \text{sgn}(t) \tan\tfrac{\pi\alpha}{2}\right)\right\},$$

where a, b (often written β), γ, and c, are real constants with $c \geq 0$, $|b| < 1$, and $0 < \alpha \leq 2$. Here, α is called the characteristic exponent. The parameters γand c determine only the location and scale; therefore, without loss of generality, we may assume $\gamma = 0$ and $c = 1$.

A stable random variable is positive-valued (respectively negative-valued) whenever $0 < \alpha < 1$ and $\beta = -1$ (respectively $\beta = 1$) in the characteristic function representation. A stable random variable with $\alpha = 2$ is a normal random variable.

SEMI-STABLE DISTRIBUTIONS

A distribution function G is said to be semi-stable if it is either normal or if the

characteristic function $\phi(t)$ of G is of the form with spectral function

$$H(-x) = x^{-\alpha} q_1(\log x),\ x > 0 \quad \text{and} \quad H(x) = -x^{-\alpha} q_2(\log x),\ x > 0$$

where $0 < \alpha < 2$, and where q_1 and q_2 are periodic functions with a common period. These functions satisfy, for all x and all $h \geq 0$,

$$e^{\alpha h} q_i(x-h) - e^{-\alpha h} q_i(x+h) \geq 0 \quad \text{and } d_i \geq q_i(x) \geq c_i, \quad i = 1,2$$

with $c_1 + c_2 > 0$.

DOMAIN OF ATTRACTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random

variables with a common distribution function F. Let $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. Set $Z_n = \frac{S_n - A_n}{B_n}$, where $\{A_n\}$ and $\{B_n\}$ are sequences of real constants with $B_n > 0$. If the sequence of normalised sums $\{Z_n\}$ converges in distribution to a random variable whose distribution function is G, then F is said to belong to the domain of attraction of G. It is well known that such a limit law is always a stable law (see Gnedenko, B. V. and Kolmogorov, A. N. [1954], p. 162).

DOMAIN OF PARTIAL ATTRACTION

Sometimes it may happen that the sequence (Z_n) does not converge for any choice of the constants (A_n) and (B_n). However, for some subsequence (n_k), the corresponding normalised sequence may converge to a non-degenerate distribution. In such a case, the distribution function F is said to belong to the domain of partial attraction of that limit law. The limit law in this setting is always an infinitely divisible distribution.

DOMAIN OF NORMAL ATTRACTION

A distribution function F (or a random variable X with distribution function F) is said to belong to the domain of normal attraction of a stable law with characteristic exponent α,

$0 < \alpha \leq 2$, if it belongs to its domain of attraction with $B_n = an^{1/\alpha}$, where a is some positive constant.

Now set $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. If the sequence $\{Z_n, n \geq 1\}$, for suitable choices of $\{A_n\}$ and $\{B_n\}$, converges weakly, then it is well known that the limit law must be stable (see Gnedenko, B. V. and Kolmogorov, A. N. [1954], p. 162). In this case, F is said to belong to the domain of attraction of the limiting stable law.

It is also known that every stable law belongs to its own domain of attraction. For an integer subsequence (n_k), and for appropriate choices of A_{n_k} and B_{n_k}, if

$$Z_{n_k} = \frac{S_{n_k} - A_{n_k}}{B_{n_k}}$$

converges weakly to a non-degenerate law, then the limit must necessarily be an infinitely divisible distribution. In this situation, F is said to belong to the domain of partial attraction of the limiting infinitely divisible law. It is true that every infinitely divisible law has a non-empty domain of partial attraction (see Gnedenko, B. V. and Kolmogorov, A. N. [1954], p. 184). Kruglov [1972] characterised the class U of limit laws of $Z_{n_k} = \frac{S_{n_k} - A_{n_k}}{B_{n_k}}$ under the assumptions that the subsequence (n_k) is strictly increasing and that $\frac{B_{n_k}}{n_k^{1/\alpha}}$ converges to a constant. They established that the class U of limit laws coincides with the class of all semi-stable laws.

When the random variables (X_n) are independently distributed with a common symmetric distribution function, Chover [1966] obtained a law of the iterated logarithm by using power normalization for the sequence of partial sums $\{S_n\}$. He established these results based on the tail probabilities of the random variable X_1. Specifically, Chover observed that for stable random variables, an LIL involving the limit superior cannot be obtained under linear normalisation, but it becomes possible under power normalisation. In fact, when the X_n's are i.i.d. symmetric stable random variables, Chover (1966) established an LIL for S_n using power normalisation; that is,

$$\lim_{n \to \infty} \operatorname{Sup} \frac{|S_n|}{n^{1/\alpha} (\log\log n)^{1/\alpha}} = C \text{ a.s.}$$

for a suitable constant C.

Later, Vasudeva (1984) extended the result to random variables in the domain of attraction of a stable law with index $0 < \alpha < 2$. Divanji and Vasudeva (1989) further extended the result to the domain of partial attraction of a semi-stable law with index $0 < \alpha < 2$.

In fact, Professor R. Vasudeva and we studied laws of the iterated logarithm for delayed sums, weighted sums, stable subordinators, moving averages, random sums, Wiener processes, and related structures and published many papers in national and international journals.

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